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SYMMETRIC POSITIVE SEMI-DEFINITE SOLUTIONS OF $AX = B$ AND $XC = D$

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Abstract

In this paper, a sufficient and necessary condition for the matrix equations $AX = B$ and $XC = D$, where $A \in \mathbf{R}^{m\times n}, B \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{n\times p}$, and $D \in \mathbf{R}^{n\times p}$, to have a common symmetric positive semi-definite solution X is established, and if it exists, a representation of the solution set S_X is given. An optimal approximation between a given matrix $\widetilde{X} \in \mathbf{R}^{n\times n}$ and the affine subspace S_X is discussed, an explicit formula for the unique optimal approximation solution is presented, and a numerical example is provided.

1. Introduction

In this paper, we shall adopt the following notation. $\mathbf{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, $\mathbf{SPR}^{n\times n}$ denotes the set of all symmetric positive semi-definite matrices in $\mathbf{R}^{n\times n}$. I_n represents the identity matrix of size n. A^T, A^+, and $\|A\|$ stand for the transpose, the

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Moore-Penrose generalized inverse, and the Frobenius norm of a real matrix A, respectively. For $A, B \in \mathbf{R}^{m\times n}$, we define an inner product in $\mathbf{R}^{m\times n}$: $\langle A, B\rangle = \text{trace}(B^T A)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. We write $A \geq 0$, if A is a real symmetric positive semi-definite matrix.

Matrix equation is one of the important study fields of linear algebra. The linear matrix equations

$$AX = B,\ XC = D, \tag{1}$$

have been considered by many authors. In [6], Mitra gave the common solution of minimum possible rank based on the generalized matrix inverses and the matrix rank. Li [4] discussed the generalized reflexive solution of (1), a necessary and sufficient condition for the solvability and the expression of the general solution were obtained. Qiu [7] considered the constraint $PX = \pm XP$ solution of (1), where P is a given Hermitian matrix satisfying $P^2 = I_n$. Qiu [8] further studied the least-squares solutions to the Equations (1) with some constraints: orthogonality, symmetric orthogonality, symmetric idempotent. Li [5] found a sufficient and necessary condition for the matrix Equations (1) to have symmetric and skew-antisymmetric solutions over the real quaternion and, for the consistent case, provided a representation of its general solution. Wang [9] considered the bisymmetric solutions of (1) over the real quaternion algebra. Recently, Dajić [2] studied the positive solutions to the Equations (1) for Hilbert space operators using generalized inverses, and a sufficient and necessary condition for its solvability, and a representation of its general solution was also established therein. In the present paper, we will consider symmetric positive semi-definite solutions of the matrix Equations (1), where $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{n\times p}$, and $D \in \mathbf{R}^{n\times p}$, and an associated optimal approximation problem:

$$\min_{X\in S_X}\|X - \widetilde{X}\|, \tag{2}$$

where $\widetilde{X}$ is a given matrix in $\mathbf{R}^{n\times n}$ and S_X is the solution set of the matrix Equations (1). Clearly, when $\widetilde{X} = 0$, the solution of (2) is the minimum norm solution of (1).

Using the singular value decomposition, we give a necessary and sufficient condition for the Equations (1) to have a solution $X \in \mathbf{SPR}^{n\times n}$, and construct the solution set S_X explicitly, when it is nonempty. We show that there exists a unique solution to the matrix optimal approximation problem (2), if the set S_X is nonempty and present an explicit formula for the unique solution.

2. The Solution of the Matrix Equations (1)

To begin with, we introduce a lemma [10].

Lemma 1. *Let* $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{m\times n}$, *and the singular value decomposition of A be*

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^T, \tag{3}$$

where $U = [U_1, U_2], V = [V_1, V_2]$ *are all orthogonal matrices and the partitions are compatible with the size of* $\Sigma = \mathrm{diag}\{\sigma_1, \cdots, \sigma_t\} > 0$, $t = rank(A)$. *Then, the matrix equation*

$$AX = B, \tag{4}$$

has a solution $X \in \mathbf{SPR}^{n\times n}$, *if and only if*

$$BA^T \geq 0,\ rank(B) = rank(BA^T). \tag{5}$$

In which case, the general solution of the equation of (4) *can be expressed as*

$$X = X_0 + V_2GV_2^T, \tag{6}$$

where

$$X_0 = A^+B + (I_n - A^+A)(A^+B)^T + (I_n - A^+A)B^T(AB^T)^+B(I_n - A^+A), \tag{7}$$

and G is an arbitrary symmetric positive semi-definite matrix.

Inserting (6) into $XC = D$, we get

$$V_2GV_2^TC = D - X_0C. \tag{8}$$

It is easily seen that the equation of (8) is equivalent to

$$V_1^T(D - X_0C) = 0, \tag{9}$$

$$GV_2^TC = V_2^T(D - X_0C). \tag{10}$$

Since, $V_1V_1^T = A^+A$ and $AX_0 = B$, then the relation of (9) is equivalent to

$$AD = BC. \tag{11}$$

Let the singular value decomposition of V_2^TC be

$$V_2^TC = P\begin{bmatrix}\Omega & 0\\ 0 & 0\end{bmatrix}Q^T, \tag{12}$$

where $P = [P_1, P_2]$, $Q = [Q_1, Q_2]$ are all orthogonal matrices and the partitions are compatible with the size of $\Omega = \mathrm{diag}\{\omega_1, \cdots, \omega_s\} > 0$, $s = \mathrm{rank}(V_2^TC)$. It follows from Lemma 1 that the equation of (10) has a solution $G \in \mathbf{SPR}^{(n-t)\times(n-t)}$, if and only if

$$C^TV_2V_2^T(D - X_0C) \geq 0,\ \mathrm{rank}((D - X_0C)^TV_2) = \mathrm{rank}((D - X_0C)^TV_2V_2^TC). \tag{13}$$

Notice that

$$V_2V_2^T(D - X_0C) = (I_n - A^+A)(D - X_0C) = D - X_0C.$$

Therefore, the relations of (13) can be simplified as

$$C^T(D - X_0C) \geq 0,\ \mathrm{rank}((D - X_0C)^T) = \mathrm{rank}((D - X_0C)^TC). \tag{14}$$

When the conditions (14) hold, the general solution of (10) can be written as

$$G = G_0 + P_2 W P_2^T,$$

where

$$G_0 = \widetilde{D}\widetilde{C}^+ + (\widetilde{D}\widetilde{C}^+)^T (I_{n-t} - \widetilde{C}\widetilde{C}^+) + (I_{n-t} - \widetilde{C}\widetilde{C}^+)\widetilde{D}(\widetilde{D}^T\widetilde{C})^+ \widetilde{D}^T (I_{n-t} - \widetilde{C}\widetilde{C}^+), \tag{15}$$

$\widetilde{C} = V_2^T C$, $\widetilde{D} = V_2^T (D - X_0 C)$, and $W \geq 0$ is an arbitrary matrix.

As a summary of the above discussion, we have proved the following result.

Theorem 1. *Given* $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{n\times p}$, *and* $D \in \mathbf{R}^{n\times p}$. *Let the singular value decompositions of A and* $V_2^T C$ *be given by* (3) *and* (12), *respectively. Then, the matrix Equations* (1) *have a solution* $X \in \mathbf{SPR}^{n\times n}$, *if and only if the conditions* (5), (11), *and* (14) *are satisfied. In this case, the solution set of* (1) *can be expressed as*

$$S_X = \{X \in \mathbf{SPR}^{n\times n} \mid X = X_1 + V_2 P_2 W P_2^T V_2^T\}, \tag{16}$$

where $X_1 = X_0 + V_2 G_0 V_2^T$, *and W is an arbitrary symmetric positive semi-definite matrix.*

3. The Solution of the Optimal Approximation Problem (2)

In order to solve the optimal approximation problem (2), we need the following lemma [3].

Lemma 2. *Let* $E \in \mathbf{R}^{n\times n}$, $E_1 = \frac{1}{2}(E + E^T)$, *and let the polar decomposition of* E_1 *be* $E_1 = FH$, *where F is an orthogonal matrix and H is a symmetric positive semi-definite matrix. Let* $\hat{E} = \frac{1}{2}(E_1 + H)$, *then*

$$\|\hat{E} - E\| = \min_{K \in \mathbf{SPR}^{n\times n}} \|K - E\|.$$

Theorem 2. *If the solution set* S_X *is nonempty, then the optimal approximation problem* (2) *has a unique solution* $\hat{X} \in S_X$. *Furthermore, let* $J = \frac{1}{2} P_2^T V_2^T (\tilde{X} + \tilde{X}^T) V_2 P_2 - P_2^T V_2^T X_1 V_2 P_2$ *and the polar decomposition of J be*

$$J = LM, \tag{17}$$

where L is an orthogonal matrix and $M \geq 0$. *Let* $\hat{W} = \frac{1}{2}(J + M)$, *then the unique solution* $\hat{X}$ *of* (2) *can be expressed as*

$$\hat{X} = X_1 + V_2 P_2 \hat{W} P_2^T V_2^T. \tag{18}$$

Proof. By Theorem 1, we know that if the conditions (5), (11), and (14) are satisfied, then the solution set S_X is nonempty. It is easy to verify that S_X is a closed convex set in Hilbert space $\mathbf{SPR}^{n \times n}$. Therefore, for a given matrix $\hat{X} \in \mathbf{R}^{n \times n}$, it follows from the best approximation theorem (see Aubin [1]), that there exists a unique solution $\hat{X}$ in S_X such that $\|\hat{X} - \tilde{X}\| = \min_{X \in S_X} \|X - \tilde{X}\|$. For any matrix $X \in S_X$, we have

$$\begin{aligned}
\|X - \tilde{X}\|^2 &= \|V_2 P_2 W P_2^T V_2^T - (\tilde{X} - X_1)\|^2 \\
&= \|V^T V_2 P_2 W P_2^T V_2^T V - V^T (\tilde{X} - X_1) V\|^2 \\
&= \xi + \|P_2 W P_2^T - V_2^T (\tilde{X} - X_1) V_2\|^2 \\
&= \xi + \|P^T P_2 W P_2^T P - P^T V_2^T (\tilde{X} - X_1) V_2 P\|^2 \\
&= \pi + \|W - P_2^T V_2^T (\tilde{X} - X_1) V_2 P_2\|^2,
\end{aligned}$$

where

$$\xi = \|V_1^T (\tilde{X} - X_1)\|^2 + \|V_2^T (\tilde{X} - X_1) V_1\|^2,$$

$$\pi = \xi + \|P_1^T V_2^T (\tilde{X} - X_1) V_2\|^2 + \|P_2^T V_2^T (\tilde{X} - X_1) V_2 P_1\|^2.$$

Therefore, $\|X - \widetilde{X}\| = \min$, if and only if

$$\|W - P_2^T V_2^T (\widetilde{X} - X_1) V_2 P_2\| = \min, \text{ s. t. } W \geq 0. \tag{19}$$

By Lemma 2, we conclude that the solution of the minimization problem (19) is $W = \hat{W}$. Substitution $W = \hat{W}$ into (16) yields (18).

4. A Numerical Example

Based on Theorems 1 and 2, we can state the following algorithm.

Algorithm 1. (An algorithm for solving the optimal approximation problem (2))

1. Input A, B, C, D, and $\widetilde{X}$.

2. If the condition (5) is satisfied, then we continue. Otherwise, we stop.

3. Find the singular value decompositions of A and $V_2^T C$ according to (3) and (12), respectively.

4. If the conditions (11) and (14) are satisfied, then the solution set S_X is nonempty and we continue. Otherwise, we stop.

5. Compute X_0 and G_0 by (7) and (15), respectively.

6. Compute $X_1 = X_0 + V_2 G_0 V_2^T$.

7. Compute the polar decomposition of the matrix $J = \frac{1}{2} P_2^T V_2^T (\widetilde{X} + \widetilde{X}^T) V_2 P_2 - P_2^T V_2^T X_1 V_2 P_2$ by (17).

8. Compute $\hat{W} = \frac{1}{2}(J + M)$.

9. Compute $\hat{X}$ according to (18).

Let $m = 2$, $n = 6$, $p = 3$. Given

$$A = \begin{bmatrix} 1.7597 & 2.7333 & 3.357 & 0.5351 & 2.4288 & 1.4819 \\ 3.7335 & 0.8502 & 2.5151 & 0.8285 & 2.5196 & 2.3006 \end{bmatrix},$$

$$B = \begin{bmatrix} 305.68 & 370.07 & 406.15 & 250.34 & 406.47 & 278.38 \\ 304.91 & 355.94 & 403.06 & 252.24 & 392.31 & 269.31 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.7085 & 4.0987 & 0.0981 \\ 0.2634 & 0.5571 & 1.1404 \\ 0.1631 & 0.212 & 3.5215 \\ 1.8761 & 3.6744 & 0.3455 \\ 0.0772 & 3.6512 & 2.2054 \\ 2.3038 & 0.0946 & 3.7887 \end{bmatrix}, D = \begin{bmatrix} 158.78 & 301.26 & 250.36 \\ 176.54 & 345.8 & 304.57 \\ 195.4 & 346.73 & 380.76 \\ 131.38 & 243.96 & 210.78 \\ 189.08 & 356.9 & 357.92 \\ 125.92 & 228.65 & 260.85 \end{bmatrix},$$

and

$$\widetilde{X} = \begin{bmatrix} 33.729 & 16.204 & 32.725 & 14.565 & 4.931 & 0.54244 \\ 8.2055 & 0.65675 & 26.206 & 31.724 & 7.198 & 26.511 \\ 21.543 & 29.16 & 6.2572 & 2.0554 & 7.0546 & 15.801 \\ 17.252 & 15.787 & 14.402 & 12.527 & 21.435 & 33.08 \\ 31.641 & 21.848 & 33.209 & 28.867 & 9.6624 & 16.543 \\ 27.055 & 28.114 & 32.55 & 0.35003 & 7.0581 & 14.862 \end{bmatrix}.$$

It is easy to verify that the conditions (5), (11), and (14) hold. Using Algorithm 1, we obtain the optimal approximation solution of (2) as follows.

$$\hat{X} = \begin{bmatrix} 25.752 & 27.237 & 25.879 & 20.869 & 26.531 & 15.815 \\ 27.237 & 38.34 & 26.119 & 19.901 & 36.213 & 20.963 \\ 25.879 & 26.119 & 44.946 & 23.629 & 34.786 & 27.803 \\ 20.869 & 19.901 & 23.629 & 17.7 & 20.803 & 13.428 \\ 26.531 & 36.213 & 34.786 & 20.803 & 38.826 & 26.043 \\ 15.815 & 20.963 & 27.803 & 13.428 & 26.043 & 19.896 \end{bmatrix}.$$

It is easily seen that $\hat{X}$ is a symmetric positive semi-definite matrix. Furthermore, we can figure out

$$\|A\hat{X} - B\| + \|\hat{X}C - D\| = 1.225e - 012.$$

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